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Golden Ratio and Fibonacci sequence in Pentagonal Constructions of Medieval Architecture

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Abstract: *The construction of the regular pentagon has always meant a difficult geometrical exercise for architects during the Middle Ages. As the correct drafting was forgotten after the Antiquity, several methods for its approximation had been invented in medieval times. As Golden Ratio appears between several parts of the regular pentagon, the role of the Fibonacci sequence in these approximate constructions is to be examined. The pentagonal drawing in the sketchbook of Villard de Honnecourt calls our attention to a possible way how medieval architects could have applied simple numerical ratios for getting angles they needed. The approximation of 72°, for instance, is likely to have been crucial for this pentagonal construction, as well as the approximation of Golden Ratio that could have been achieved by neighbouring pairs from Fibonacci's sequence.*
Keywords: *Fibonacci, medieval, architecture, Golden Ratio, pentagon.*

1. INTRODUCTION

The question whether Golden Ratio had been used by architects of the Middle Ages was first raised in the Hungarian literature by Viktor Miskovszky in 1878 by the examination of the proportions of the tabernacle of the Basilica of Saint Giles in Bardejov (Bártfa). [1] Though his argument based on the comparison of accurately measured dimensions seems quite convincible, we need to find further proofs of the medieval application of the Golden Ratio supported by mathematical background knowledge to resolve this riddle. In the international literature a famous supporter of the idea of the use of Golden Ratio and in gothic cathedral constructions was Frederik Macody Lund. [2] While his theory was widely rejected, discussion on the presence of the Golden Ratio as well as pentagonal constructions in medieval architecture has still been topical in recent research.

While the scientific culture of the Antiquity has continued to flourish in the Byzantine Empire, the majority of this knowledge had been forgotten in Western Europe.

The great Migration Period has opened an entirely new chapter in the history of science with a fundamental change of perspective. While the solidification of the new religion had taken centuries, the ethos of Early Christian culture tended to determine people's thinking totally. While ancient Greeks and Romans have elaborated a highly sophisticated theory-based system of natural sciences, the western societies of the early Middle Ages only have developed the technical knowledge they directly needed and applied for their everyday life.

It is mostly due to the quickening of trade with the East and the Arabic culture that scientific interest of Western Europe has wakened. The oeuvre of the Italian merchant and mathematician Leonardo di Pisa (Fibonacci) certainly meant a great divide in the scientific culture and people's approach to science, especially arithmetic and geometry among the seven liberal arts. *Liber Abaci* including the famous problematic of the rabbits concluding the Fibonacci sequence (in which the quotient of the neighbouring members converges the Golden Ratio) might have been widely spread and known. [3] Its importance can be demonstrated by the approximately dozen surviving copies dating back from the 13th to the 15th century. [4] Besides *Liber Abaci*, Fibonacci's other works are also of high interest, especially *De Practica Geometriae*, that contains practical exercises, measurements and surface calculus of the pentagon. (Barnabas Hughes also has claimed that the goal of the book perhaps had been to *„…fire the imagination of builders with analyses of pentagons and decagons.*") [5]

Both the Fibonacci sequence and the pentagons of *De Practica Geometriae* have meant serious influence in the revival of the Golden Ratio in the 13th century, which must have effected innovations in architecture as well. By the examination of the medieval problematic of pentagonal designs new information on Fibonacci's direct impact on architects can be detected.

2. HISTORY AND NATURE OF THE GOLDEN RATIO

Among all regular polygons, the pentagon and the decagon are the ones in the construction of which the Golden Ratio (*Sectio Aurea*) appears. The method of the golden division of a section had already been known in the ancient times. [6] While numerous construction methods could be mentioned, the theorem where the product of the secants drawn from an external point to a circle is constant (tangent-secant theorem), had already been known in the second century BC. [7] As a result of the construction applying this theorem, section a is to be divided into sections x and $a - x$ with their quotient of the Golden Ratio.

Thus comes the $x^2 - ax = a^2$ the radical of which (the one relevant from the aspect of our case) is $\frac{1+\sqrt{5}}{2}$, called Golden Ratio and signed by ϕ.

In the construction of a regular pentagon, fine interlace of geometric and algebraic results can be admired. All the great knowledge of the Pythagoreans meets in their symbol, a pentagram, which is called *Pentagramma*, inscribed into a pentagon. In the fourth volume of *Elements* Euclid has concluded the construction of the regular pentagon and decagon through several exercises. He has demonstrated how the isosceles triangle, whose base angle is two times the vertex angle, could be constructed. Such triangle provides the side of a regular pentagon whose circumscribed circle is equivalent to the circumscribed circle of this triangle. Since his method of pentagon construction had not spread, during the Middle Ages as well as in later times several approximations and accurate constructing methods appeared including some really ingenious ones. In 1202, in his *Liber Abaci*, apropos of the exercise about the reproduction of the rabbits, Fibonacci mentioned his sequence that had already been worked out earlier in the Ancient Hindu Pingala's *Chandaḥśāstra* around 200 BC. [7]

Although it carries his name, the currently known first written documentation of the sequence is fifty years older than *Liber Abaci*. (Acharya Hemachandra's *Chandonushasana* around 1150.) [7] The limit of the sequence produced by the quotients of the members of the well-known recursive Fibonacci sequence ($F_{n+2} := F_{n+1} + F_n,\ \ F_0 = 1,\ F_1 = 1$) is equal to the Golden Ratio, that is $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$. (Figure 1)

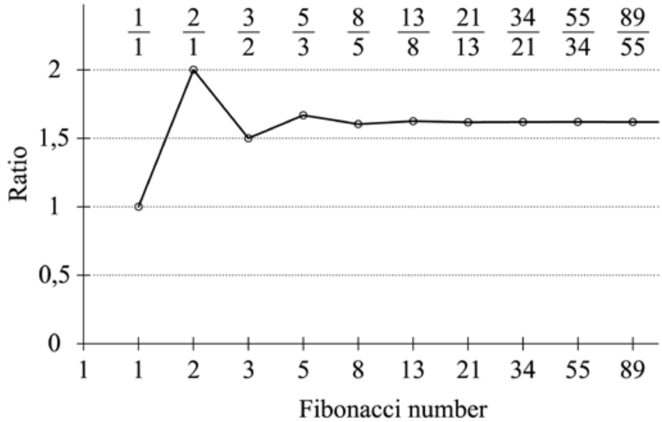

Figure 1. Values of the ratio of neighbouring members of the Fibonacci section (Authors' diagram)

For the different appearances of the Golden Ratio in a regular pentagon the following figure should be examined. (Figure 2)

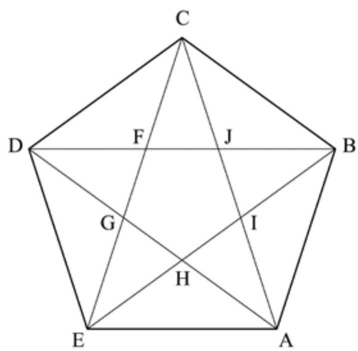

Figure 2. Regular pentagon (Authors' drawing)

- The proportion of the sides (e.g. *AB*) and the diagonals (e.g. *AC*) of a regular pentagon is ϕ.
- The diagonals intersect each other with the proportion of the Golden Ratio (e.g. *HC*:*AH*= ϕ).
- The proportion of the sides of the original pentagon (e.g. *AB*) and those of the smaller pentagon produced by the diagonals (e.g. *HI*) is ϕ.

The regular pentagon construction of Euclid applies these coherences.

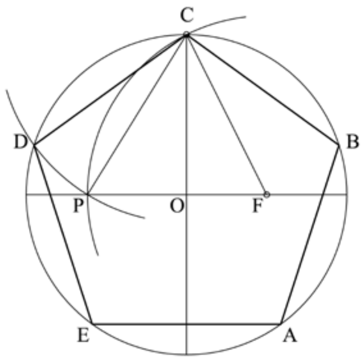

Figure 3. Precise construction of the regular pentagon (Authors' drawing)

It is probable that during the Middle Ages the Golden Ratio could have been used for some kind of approximation of the pentagon construction, as László Hoppe has suggested. [8] (Figure 4) It must be emphasised however that no medieval sources testify the actual application of such methods. Following a similar logic to the one of the regular drafting discussed by Euclid, (Figure 3) Hoppe suggested two speculative methods using neighbouring members of the Fibonacci sequence for getting the side of the pentagon from the radius of the circumcircle. These neighbouring pairs are practically the 5:8 and 8:13, the constructions of which could have been rather simple for architects.

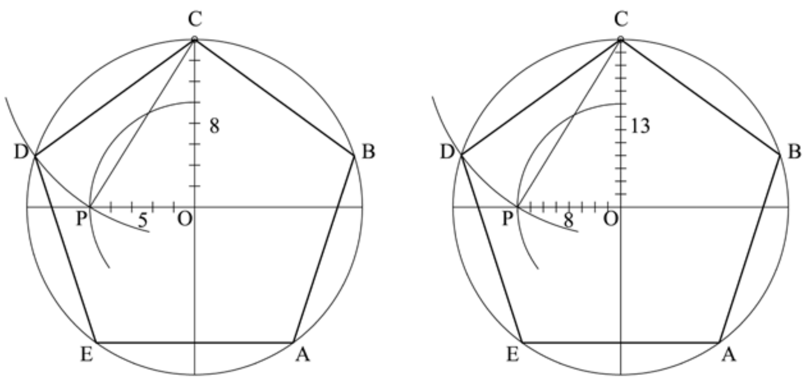

Figure 4. Two variations for the approximation of the pentagon using neighbouring members of the Fibonacci sequence (Authors' drawing by [8])

It is generally known that in an n-sided regular polygon the relation between the radius of the circumcircle (R) and the side (a) is $\frac{R}{a} = \frac{1}{2\sin\frac{\pi}{n}}$ that is $\frac{R}{a} = \frac{1}{2\sin 36°}$ in the case of a pentagon. So what is the relation between $\frac{R}{a} = \frac{1}{2\sin 36°}$ and Golden Ratio? Let's take the radicals of the solutions of the above mentioned $x^2 - ax = a^2$ second-degree equation, which are $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$. The prior is the Golden Ratio signed by ϕ, while the ulterior is generally signed by φ.

It's noticeable that $\phi\varphi = 1$ and $\phi + \varphi = \sqrt{5}$.

Shall we go back to $\frac{R}{a}$ ratio!

Since $\sin 36° = \frac{\sqrt{5\frac{\sqrt{5}-1}{2}}}{2} = \frac{\sqrt{5\varphi}}{2} = \frac{1}{2}\sqrt{\frac{5}{\phi}}$

hence $\frac{R}{a} = \sqrt{\frac{\phi}{5}} = \frac{1}{\sqrt{5\varphi}}$.

How does this appear in the geometrical approximations of the pentagon? (Figure 4)

There, for the side of the pentagon would be $a = \sqrt{1+\left(\frac{5}{8}\right)^2}\, R$ or other $a = \sqrt{1+\left(\frac{8}{13}\right)^2}\, R$.

Shall we evaluate $\frac{5}{8}$ and $\frac{8}{13}$ with $\breve{\varphi}$ which is approximated $\left(\breve{\phi} = \frac{1}{\breve{\varphi}}\right)$.

$$\frac{R}{a} = \frac{1}{\sqrt{1+\breve{\varphi}^2}} = \frac{1}{\sqrt{\breve{\varphi}\breve{\phi}+\breve{\varphi}^2}} = \frac{1}{\sqrt{\breve{\varphi}(\breve{\phi}+\breve{\varphi})}} = \frac{1}{\sqrt{5\breve{\varphi}}} = \sqrt{\frac{\breve{\phi}}{5}}$$

Thus the accuracy of these approximations can be counted (Table 1)

$\phi \approx \frac{8}{5}$	99,441158 %
$\phi \approx \frac{13}{8}$	100,215030 %
$\phi \approx \frac{21}{13}$	99,918096 %
$\phi \approx \frac{34}{21}$	100,031318 %

Table 1. Accuracy of pentagon approximations using neighbouring pairs of the Fibonacci sequence

This concludes that the approximations that Hoppe has suggested are very close to the regular pentagon. Hoppe's suggestion is not the only simple way how medieval master masons might have taken advantage of the Golden Ratio in the approximation of the pentagon. As Nigel Hiscock and Tomás Gil-López [9, 10] have mentioned, the golden triangle (whose base and legs are in Golden Ratio) could have been known and applied by medieval architects. A similar isosceles triangle can be found in a regular decagon where the base is equal to the side and the legs are equal to the radius of the circumcircle of the decagon. In the golden triangle the base angles are 72° and the vertex is 36°. (Figure 5)

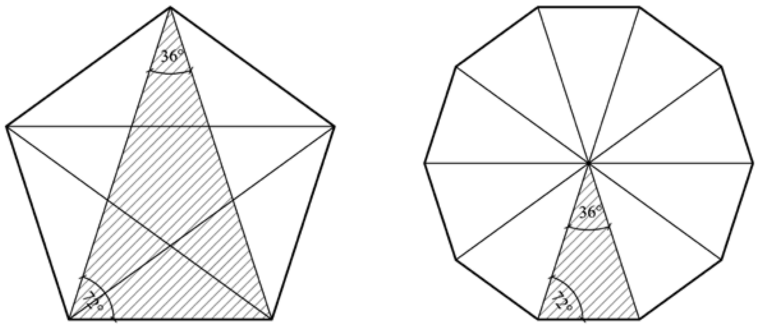

Figure 5. Example of the golden triangle in a pentagon and a decagon (Authors' drawing)

3. PENTAGON CONSTRUCTION IN VILLARD DE HONNECOURT'S SKETCHBOOK

The problematic of the geometrical construction of the pentagon occurs in several medieval sources such as Mathias Roriczer's *Geometria Deutsch*, the *Musterbuch of Hans Hammer*, or two sketches in the medieval plan collection of the Akademie der Bildenden Künste Wien. [8] However the most cited and best known pentagonal figure is the drawing of a tower of five edges in folio 21 recto [11] of the 13th-century *Portfolio of Villard de Honnecourt*. (The text written below the drawing in question – *'par chu portrait om one / toor a chinc arestes'* – indicates that its purpose has been the presentation of a pentagonal tower: *'By this* [means] *one represents a tower with five edges.'*)

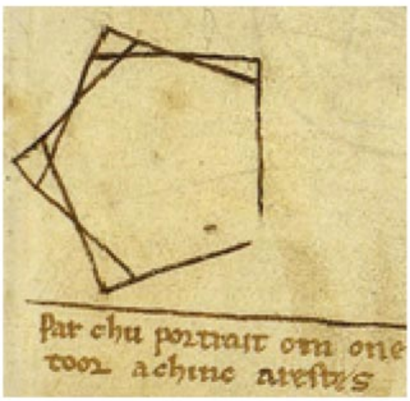

Figure 6. Drawing of the tower of five edges in the Portfolio of Villard de Honnecourt (After [11])

Examining this sketch, it must be taken into consideration that one successor of Villard, called *Hand IV* by Barnes [11] has erased the original drawing from the previous palimpsest page and redrawn it. (Figure 6). Robert Branner and Roland Bechmann [12, 13] have detected and reconstructed the traces of the original pentagon (presumably drafted by Villard) applying ultraviolet lighting, which implies some differences compared to the redrawn figure of Hand IV. This latter seems to confirm the suggestion of Barnes that Hand IV could hardly have been an architect as he has redrawn several other drawings without understanding their principles. [11] Although the original logic of the geometrical construction of the pentagon can still be identified, the whole drafting had not been copied but aborted before the last steps.

The key of the construction was reconstructed by Cord Meckseper [14] and then in the same way by Bechmann [13] independent from the former. (Meckseper's opinion was shared by László Hoppe. [8]) Their idea for the reconstruction of the approximation of the regular pentagon has been the application of right triangles of legs in a 1 to 3 ratio, or in other words it has been the rotation of right angles so that their intersection had resulted in a 1 to 3 ratio of their legs. (Figure 7) All the scholars who have mentioned and accepted this logic of the construction have shared the opinion that the drafting had been very simple by using the framing square with one and three units on its both legs. Meckseper and Hoppe suggested that the 1 to 3 ratio served the approximation of the 72° angle (as $3 \approx \tan 71{,}5651°$) which is the supplementary angle of the 108° internal angle of the regular pentagon. (Werner Müller has counted that the mathematical accuracy of the approximation is abundantly acceptable for architectural drawing. [15])

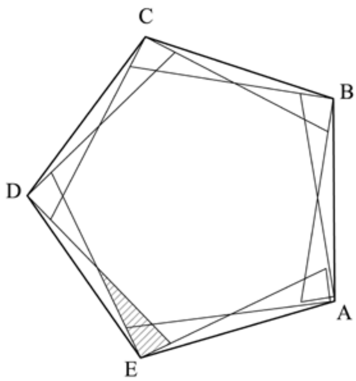

Figure 7. Explanation of Villard's drawing suggested by Meckseper and Bechmann (Authors' drawing after [14] and [13])

However the construction that Meckseper and Bechmann have suggested is certainly convincing, it yet represents some contradictions with the drawing of Hand IV. It is worth noticing that the bottom right angle of Hand IV's pentagon (*A* in Figure 8) seems to be properly the 108° internal angle of the pentagon. (This anomaly of the figure has also been detected by Bechmann. [13]) Considering this, Figure 8 shows a different position of the pentagon in the tower drawing without suggesting that this could be the original logic of the figure, as that being a rough sketch, the 108° angle of vertex *A* (Figure 8) can easily be accidental.

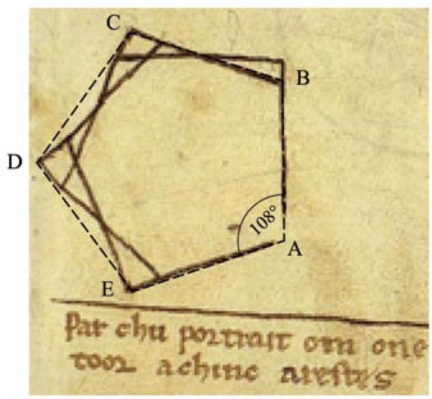

Figure 8. An alternative position of the pentagon in the tower drawing applying that the angle at vertex *A* is haphazardly 108° (Authors' drawing based on [11])

Accenting that the schematic feature of Hand IV's drawing allows a wide scale of explanations, it is still arguable if the theory of Meckseper and Bechmann would be the closest to the original traces. Although vertex *C* and *D* in both Figure 8 and 9 fit to the tower drawing, the other steps of the suggested construction (producing vertex *A*, *B* and *E* in Figure 7 and 9) are distorted. (Figure 9)

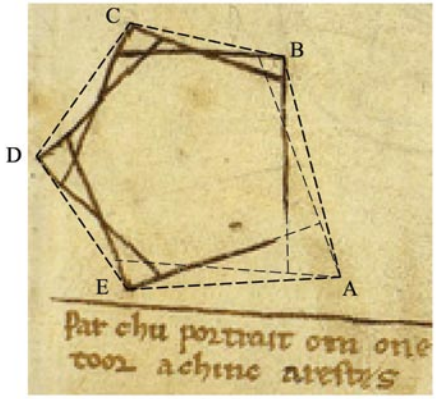

Figure 9. The system suggested by Meckseper and Bechmann projected to the original drawing (Authors' drawing based on [11])

An alternative explanation can be suggested, whose logic differs from the one of Meckseper and Bechmann, but pursues more tightly the original traces. Although Meckseper himself has also emphasised the connection between the right triangle of legs of 1 to 3 ratio and the approximation of the $\tan 18°$ (or $\tan 72°$), in his reconstruction of the method this coincidence is not directly exploited. The steps Figure 10 drafts, however, are directly based on the approximation of the 72° angle indeed. According to this suggestion, considering segment AB as the initial side and supposing a heading from right to left, a right triangle of legs of 1 to 3 ratio is to join to each prolonged previous side, so that the direction of the next side could be defined. (Figure 10) The length of the pentagon sides is to be determined by two equal right triangles in each vertex. Thus the finishing pentagon is not to be located outside the traces of the original figure (as in the system of Meckseper and Bechmann), but fitting to them. By projecting this alternative reconstruction of the pentagonal construction to the original tower drawing, a much closer correspondence can be detected, where only the last step represents an anomaly. (Figure 11)

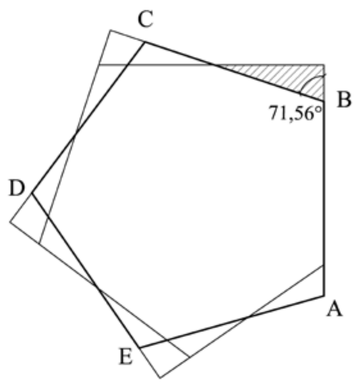

Figure 10. Alternative suggestion for the reconstruction of the pentagon construction (Authors' drawing)

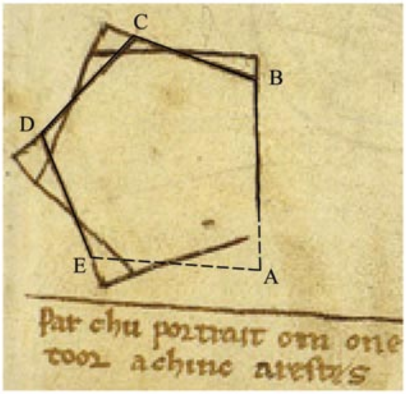

Figure 11. Alternative suggestion projected to the original drawing (Authors' drawing based on [11])

4. CONCLUSION

Several architectural examples demonstrate that the construction of the pentagon has been an important desing tool in medieval master mason's hand (the 'pillars of light' in the Saint Mary's Church in Freistadt (1484), the staircase tower of the upper level of Martinsturm of Basel Cathedral, the tower of the Clarissine Church in Bratislava and a pentagonal baldachin appended to a pillar in the Church of the Holy Spirit in Landshut (1461).) The proportions of the golden triangle that has been mentioned by Hiscock and Gil-López [9, 10] as a simple way of translation of design to site, could have been known and used by medieval architects. As in several geometrical construction methods, in the case of the drafting of the golden triangle, simple numerical ratios could be applied, that has resulted an approximation within the margin of errors of the architectural design or layout. In theory, neighbouring members of the Fibonacci sequence can be used for this approximation (for example [16]), as in the case of the two pentagon constructions suggested by Hoppe. (Figure 4) Golden triangle also can be applied for the definition of 72°, similarly to the logic involved in the pentagonal tower drawing in Villard's Portfolio. As the 1 to 3 proportion resulting the approximation of tan72° had likely been used by medieval architects. Indeed, more numerical ratios could have been known for producing the same angle, perhaps selected from the neighbouring elements of Fibonacci's sequence.

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